Logarithmic corrections in the free energy of monomer-dimer model on plane lattices with free boundaries

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Using exact computations we study the classical hard-core monomer-dimer models on $m \times n$ plane lattice strips with free boundaries. For an arbitrary number v of monomers (or vacancies), we found a logarithmic correction term in the finite-size correction of the free energy per lattice site. The coefficient of the logarithmic correction term depends on the number of monomers present (v) and the parity of the width n of the lattice strip: the coefficient equals to v when n is odd, and $v/2$ when n is even. The results are generalizations of the previous results for a single monomer in an otherwise fully packed lattice of dimers. We also study the finite-size correction in the low dimer density limit, where the number of dimers d is fixed. In this case the coefficient of the logarithmic correction term equals to d, for both odd and even n.

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I. INTRODUCTION

The monomer-dimer problem is a classical model in statistical physics. In the model the diatomic molecules are modeled as rigid dimers which occupy two adjacent sites in a regular lattice and no lattice site is covered by more than one dimer. The lattice sites that are not covered by the dimers are regarded as occupied by monomers. A central problem of the model is to enumerate the dimer configurations on the lattice. A breakthrough came in 1961 when an elegant exact closed-form solution was found for a special case of the model in plane lattices, namely when the lattice is completely covered by dimers (the close-packed dimer problem, or dimer-covering problem) [1]. The method used to achieve this solution, however, cannot be extended to tackle the more general monomer-dimer problem where there are vacancies in the lattice, and the problem remains unsolved notwithstanding years of efforts. Rigorous results exists only for series expansion in low dimer density [2], lower bounds on free energy [3], monomer-monomer correlation function of two monomers in a lattice otherwise packed with dimers [4], and locations of zeros of partition functions [5]. Some approximate methods have been proposed [6]. One recent advance is an analytic solution to the special case of the problem where there is a single vacancy at certain specific sites on the boundary of the lattice [7].

There is renewed interest in monomer-dimer problem recently. The interest comes from different directions. Besides the intrinsic interest of the problem itself and its close relation to the well-studied Ising model [8], the model also acts as the classical limit of the recently introduced quantum dimer model [9], which has been investigated intensively as the central model in modern theories of strongly correlated quantum matter.

The problem also attracts attention in the field of computational complexity. It has been shown that two-dimensional monomer-dimer problem is computationally intractable and belongs to the "*#P-complete*" class [10]. #P-complete class plays the same role for counting problems (such as counting dimer configurations discussed here) as the more familiar *NP-complete* class for the decision problems (such as the well-known traveling salesman problem). The #P-complete problems are at least as hard as the NP-complete problems [11]. If *any* problem in the #P-complete class is found to be solvable, every problem in #P class is solvable. Currently it is not clear whether there exists any such solution to the #P-complete or NP-complete class problems, and "P verse NP" problem is perhaps the major outstanding problem in theoretical computer science.

In this paper we report the logarithmic correction in the free energy of the monomer-dimer model with arbitrary number of monomers on plane lattice strips of size $m \times n$, where the width n is fixed, and address the finding in the context of universality, scaling, and exact finite-size corrections in two-dimensional critical systems. These topics have been studied intensively in recent decades. As one of the few non-trivial exactly solved statistical lattice models, the dimer model attracts much attention recently to test the predictions of conformal field theory [12] and finite-size scaling [13]. Unlike other models, the (close-packed) dimer model on a plane lattice shows certain peculiarity with respect to the predictions of the theories, both for the central charge [14,15] and for the logarithmic corrections [14]. The free energy per unit length of a two-dimensional system at criticality is assumed to have the form of [12,13]

$$F_n = nf_b + f_s + C \ln n + D/n + \cdots . \tag{1}$$

The first two terms, f_b and f_s, are the "bulk" and "surface" terms, respectively. These two terms are not universal and depend on the details of the model. The other two coefficients, C and D, are supposed to be universal, depending only on the shape of the system and the boundary conditions. The dimensionless coefficient D is related to the central charge of the system and the boundary conditions in the transversal direction [12]. The coefficient of the logarithmic term C depends on the geometry of the system. For systems with corners, as for the lattices with free boundaries dis-

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cussed here, $C=\Sigma_i u_i^{\text{corner}}$, with some universal contribution u_i^{corner} from each corner [13]:

$$u_i^{\text{corner}} = \frac{c\gamma_i}{24\pi}[1-(\pi/\gamma_i)^2].$$

Here c is the central charge and γ_i is the interior angle of the corner.

From the exact solution of the close-packed dimer problem, the asymptotic expression of the free energy $F_{m,n}=\ln a_{mn/2}$ was found to be [16]

$$F_{m,n} = mnf_b + (m+n)f_s + D_n[(m+1)/(n+1)] + o(n^{-2+\delta}) \tag{2}$$

where $\delta>0$ and D_n depends on the geometry of the lattice and the parity of n. Two issues arise from this asymptotic expression. The first issue is related to the central charge, because the expansion depends on the parity of n:

$$\frac{F_{m,n}}{mn} = \frac{G}{\pi} + \left[\frac{G}{\pi} - \frac{1}{2}\ln(1+\sqrt{2})\right]\frac{1}{n} + \frac{\pi}{24}\frac{1}{n^2} + \cdots$$

when n is even, and

$$\frac{F_{m,n}}{mn} = \frac{G}{\pi} + \left[\frac{G}{\pi} - \frac{1}{2}\ln(1+\sqrt{2})\right]\frac{1}{n} - \frac{\pi}{12}\frac{1}{n^2} + \cdots$$

when n is odd. Here G is the Catalan's constant: $G=1-3^{-2}+5^{-2}-7^{-2}+\cdots=0.915965594$. At the first look, the coefficients of n^{-2}, which are related to the central charge of the conformal field theory, are different for even and odd values of n, leading to different values of central charge c: $c=1$ for the even n and $c=-2$ for odd n. Recently this issue has been addressed using logarithmic conformal field theory [15,17]. It is pointed out that the coefficients of n^{-2} are actually related to the *effective* central charge $c_{\text{eff}}=c-24h_{\min}$, where $h_{\min}$ is the conformal weight of the ground state and is a boundary dependent quantity. By using a bijection of the dimer coverings with a spanning tree and Abelian sandpile model, it is found that changing the parity of n results in the change of the boundary conditions of the mapped spanning trees, leading to different c_{eff}. The central charge itself, however, remains unchanged $c=-2$ [15].

The second issue, which was pointed out previously [14], is the absence of the corner contribution to the logarithmic correction term in the expansion of Eq. (2) of the close-packed dimer model, as predicted by the theory [Eq. (1)] for the two-dimensional critical systems with free-boundary. Recently we found that when there is one monomer in the lattices, there is a logarithmic term in the finite-size correction of the free energy [18]. In the following (Sec. II) we generalize the results to plane lattices with arbitrary number of monomers, and find that the coefficients of the logarithmic correction term depend not only on the number of monomers, but also on the parity of n, the width of the lattice strip. As a comparison, in Sec. III we look at the low dimer limit where the number of dimers d is fixed. In this case we also find a logarithmic term in the finite-size correction of the free energy per lattice site. The coefficient of the logarithmic term, however, equals to the number of dimers d and does not depend on the parity of n, the width of the lattice. In the Appendix, we investigate the free energy and number-theoretical properties of $a_{(n^2-v)/2}$, the number of ways to arrange $(n^2-v)/2$ dimers with fixed number v of monomers on $n\times n$ square lattices, in a manner that is parallel to Ref. [18], where the cases $v=0$ and $v=1$ are reported.

II. LOGARITHMIC CORRECTIONS IN THE FREE ENERGY FOR FIXED NUMBER OF MONOMERS

The configurational grand canonical partition function of the system is

$$Z_{m,n}(x) = a_N x^N + a_{N-1}x^{N-1} + \cdots + a_0$$

where a_d is the number of ways to arrange d dimers on the $m\times n$ plane lattice with free boundaries. We are interested in the coefficients of fixed number v of monomers, that is, $a_{(mn-v)/2}$, when the width of the lattice strip n is fixed and the length of the strip m changes. When the width of the lattice strip n is even, v can only take even numbers; when n is odd, v can take either even or odd values based on the parity of m.

The full partition functions have been calculated exactly for n from 1 to 16, using the method discussed in Refs. [18,19]. In the previous study of a single monomer in otherwise close-packed lattices [18], special algorithm was used to take advantage of the fact that in order to have only one monomer in the lattices, each horizontal dimension of the lattices can have at most one monomer (we expand the lattices in the vertical direction); other configurations can be safely ignored with no effect on the coefficient of the leading term of the partition functions. This special algorithm allows for calculation on lattices with width up to $n=19$. In the current study of the general case of arbitrary number of monomers, full partition functions are calculated. To do this all valid configurations have to be considered, and the number of valid configurations is much bigger than that in the previous study. For this reason the partition functions are calculated for lattices with width n up to 16. The formulas for the size of the matrix and the number of total valid configurations are given in Ref. [18].

The coefficients $a_{(mn-v)/2}$ of the partition functions are extracted to fit the following function:

$$\frac{\ln a_{(mn-v)/2}}{mn} = c_0 + \frac{c_1}{m} + \frac{c_2}{m^2} + \frac{c_3}{m^3} + \frac{c_4}{m^4} + \frac{bv\ell}{n}\frac{\ln(m+1)}{m}. \tag{3}$$

where $b=1$ when n is odd, and $b=1/2$ when n is even. The reason to choose $\ln(m+1)$ instead of $\ln(m)$ is discussed in Ref. [18], and the choice will not affect the results discussed below.

For the close-packed dimer problem (where $v=0$ and mn is even), the exact solution gives c_0 in Eq. (3) as [1]

$$c_0^e(n) = \frac{1}{n}\ln\left[\prod_{i=1}^{n/2}\left(\cos\frac{i\pi}{n+1} + \left(1+\cos^2\frac{i\pi}{n+1}\right)^{1/2}\right)\right]. \tag{4}$$

For a given n, this exact expression of c_0 actually holds for

TABLE I. Parameter ℓ in the coefficient of the logarithmic term from fitting $(mn)^{-1}\ln a_{(mn-v)/2}$ to Eq. (3) for *odd* values of n, with $b=1$. Only data with $m\geqslant m_0=100$ are used in the fitting. For $n=1$, $m\leqslant 2000$; for $n=3,\ldots 11$, $m\leqslant 500$; for $n=13$, $m\leqslant 408$. for $n=15$, $m\leqslant 200$. Numbers in square brackets denote powers of 10.

	n							
v	1	3	5	7	9	11	13	15
0	0	−1.42355[−12]	6.9891[−10]	7.10235[−10]	1.03087[−9]	1.56222[−9]	3.76785[−9]	4.80826[−9]
1	1	1	1	1	1	1	1	1
2	1	1	1	0.999999	0.999998	0.999997	0.999993	0.999971
3	1	0.999998	0.999991	0.999975	0.999945	0.999897	0.999781	0.999161
4	1	0.999988	0.999945	0.999851	0.99969	0.999449	0.998907	0.996441
5	1	0.999962	0.999835	0.999576	0.999163	0.998585	0.99736	0.992426
6	1	0.999912	0.999632	0.999108	0.998322	0.997282	0.995194	0.987601
7	1	0.999836	0.999351	0.998494	0.997273	0.995726	0.99273	0.98253
8	1	0.999721	0.998951	0.997671	0.995927	0.993799	0.98982	0.977101
9	1	0.999581	0.998493	0.996767	0.9945	0.991812	0.986905	0.97186
10	1	0.99939	0.997902	0.995649	0.99279	0.989492	0.983621	0.966407
11	1	0.999181	0.997289	0.994527	0.991117	0.987267	0.980527	0.96134
12	0.999999	0.998909	0.996529	0.99318	0.989159	0.984713	0.977081	0.956094

any finite number v of monomers. This was confirmed by fitting the data with c_0 as a free parameter. In the following we use $c_0^e(n)$ given in Eq. (4) for c_0 in Eq. (3). We fit the data extracted from the full partition functions to Eq. (3) for $v=0,1,\ldots,12$ when n is odd: $n=1,3,\ldots,15$, and for $v=0,2,4,\ldots,24$ when n is even: $n=2,4,\ldots,16$. The fitting results for ℓ are shown in Tables I and II for odd and even values of n, respectively.

For $n=1$, the following exact result can be obtained for any m and v when they have the same parity:

$$a_{(m-v)/2} = \binom{(m+v)/2}{v} = (2^v v!)^{-1}(m+v)(m+v-2)\cdots(m-v+2),$$

which gives $\ell=1$ exactly for $v>0$. The fitting results for $n=1$ as listed in Table I are used as a check for the fitting procedure.

As we did previously [18], only data with $m\geqslant m_0=100$ are used in these fittings. The curves of fitting for two values

TABLE II. Parameter ℓ in the coefficient of the logarithmic term from fitting $(mn)^{-1}\ln a_{(mn-v)/2}$ to Eq. (3) for even values of n, with $b=1/2$. Only data with $m\geqslant m_0=100$ are used in the fitting. For $n=2$, $m\leqslant 2000$; for $n=4$, $m\leqslant 1000$, for $n=6,\ldots 12$, $m\leqslant 500$; for $n=14$, $m\leqslant 200$. for $n=16$, $m\leqslant 143$. Numbers in square brackets denote powers of 10.

	n							
v	2	4	6	8	10	12	14	16
0	1.21643[−11]	1.3329[−10]	5.89405[−10]	9.28482[−10]	9.0419[−10]	1.13781[−9]	8.97751[−8]	1.95081[−5]
2	1	1	1	0.999999	0.999999	1	1.00017	1.00159
4	1	1	1.00001	1.00002	1.00006	1.00012	1.00075	1.00063
6	1	1	1.00002	1.00003	1.00004	1	0.998332	0.992121
8	1	1	0.999994	0.999931	0.999739	0.999318	0.993311	0.982993
10	1	0.999997	0.999907	0.999639	0.999043	0.998001	0.987848	0.978479
12	1	0.999982	0.999718	0.999103	0.997943	0.996201	0.983945	0.980273
14	1	0.999951	0.9994	0.998314	0.996526	0.994189	0.982769	0.987943
16	1	0.999897	0.998938	0.997305	0.994942	0.99226	0.984712	1.00024
18	0.999997	0.999816	0.998332	0.996139	0.993357	0.990678	0.989654	1.01582
20	0.999992	0.999702	0.997596	0.994896	0.991933	0.989648	0.997209	1.03352
22	0.999983	0.999551	0.996756	0.993664	0.990808	0.989309	1.00689	1.05241
24	0.99997	0.999361	0.995843	0.992527	0.990091	0.989743	1.01822	1.07182

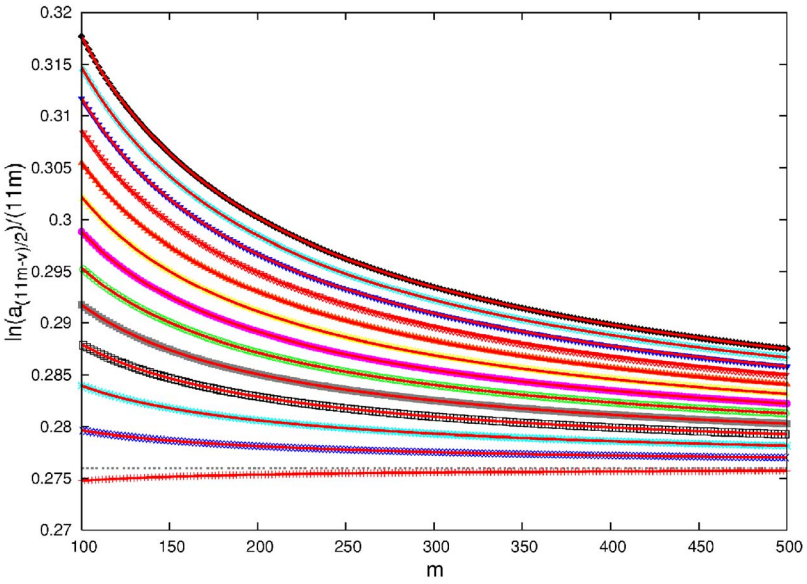

FIG. 1. (Color online) The original data of $(mn)^{-1}\ln(a_{(mn-v)/2})$ and the fitted curves for $n=11$ and $v=0,1,\ldots,12$. The curve for $v=0$ is on the bottom and that for $v=12$ is on the top. The dashed horizontal line is $c_0^e(11)=0.276\,016\,066\,62$ from the exact expression Eq. (4).

of n, $n=11$ and $n=12$, are shown in Figure 1 and Figure 2, respectively. The fitting results for all the parameters when $n=11$ and $n=12$ are shown in Tables III and IV, respectively. These results lead clearly to the conclusion that for a fixed number of monomers v, there is a logarithmic correction term in the free energy $(mn)^{-1}\ln a_{(mn-v)/2}$, and the coefficient of this term depends not only on v, but also on the parity of n, the width of the lattice strip: when n is odd, the coefficient equals to v; when n is even, the coefficient is $v/2$. The lack of the logarithmic correction term found in the close-packed dimer problem ($v=0$) [1,14,16] and the logarithmic correction term found in the odd-by-odd lattice with one single vacancy ($v=1$) [18] are just special cases of this general result.

It should be pointed out that the results are not restricted to small values of monomers (v). Numerical evidences clearly show that the results hold for any given number of monomers.

III. LOGARITHMIC CORRECTIONS IN THE FREE ENERGY FOR FIXED NUMBER OF DIMERS

In this section we study the situation where the number of dimers d is fixed on a lattice strip with a width of n, while the length m of the lattice strip goes to infinity. This low dimer density problem is usually considered as an easier problem compared with the high dimer density (low monomer density) problem discussed in the previous section.

As in the previous section for fixed number of monomers, the coefficients a_d with d dimers are extracted from the full partition function. These coefficients are then fitted to the following function:

$$\frac{\ln a_d}{mn} = c_0 + \frac{c_1}{m} + \frac{c_2}{m^2} + \frac{c_3}{m^3} + \frac{c_4}{m^4} + d\frac{\ell}{n}\frac{\ln(m+1)}{m}. \qquad (5)$$

For this situation we no longer put different factors for odd and even n.

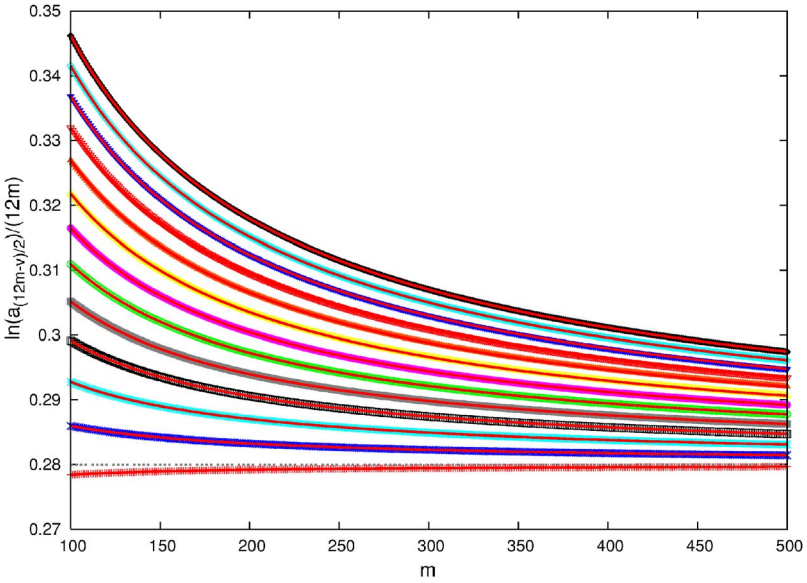

FIG. 2. (Color online) The original data of $(mn)^{-1}\ln(a_{(mn-v)/2})$ and the fitted curves for $n=12$ and $v=0,2,\ldots,24$. The curve for $v=0$ is on the bottom and that for $v=24$ is on the top. The dashed horizontal line is $c_0^e(12)=0.279\,975\,752\,03$ from the exact expression Eq. (4).

TABLE III. Fitting $(mn)^{-1}\ln a_{(mn-v)/2}$ to Eq. (3) for $n=11$ with $b=1$ and c_0 fixed as $c_0^e(11)=0.276\,016\,066\,62$ given by Eq. (4). Only data with $m_0=100\leqslant m\leqslant 500$ are used in the fitting. Numbers in square brackets denote powers of 10.

v	c_1	c_2	c_3	c_4	ℓ
0	−0.125966	8.75546[−8]	−8.24421[−6]	0.000319302	1.56222[−9]
1	−0.0561077	−0.140372	0.0319911	−0.0345653	1
2	−0.0492583	0.474695	−4.7249	23.4862	0.999997
3	−0.0790665	1.82165	−36.8528	449.079	0.999897
4	−0.133937	3.79131	−103.787	1715.28	0.999449
5	−0.207035	6.24906	−203.292	3861.49	0.998585
6	−0.293685	9.05639	−328.77	6728.3	0.997282
7	−0.391277	12.162	−480.3	10421.5	0.995726
8	−0.496769	15.4114	−645.37	14479.9	0.993799
9	−0.609818	18.8669	−833.317	19374.1	0.991812
10	−0.727328	22.305	−1022.43	24212.7	0.989492
11	−0.850844	25.9246	−1235.21	30014.7	0.987267
12	−0.976389	29.3893	−1437.02	35307.6	0.984713

As d is fixed and m goes to infinity, the dimer density approaches zero. It is clear that $c_0=0$ in Eq. (5). This can also be confirmed by fitting the data with c_0 as a free parameter. In the following fittings we set $c_0=0$.

We fit the data of $(mn)^{-1}\ln a_d$ to Eq. (5) for $d=1,2,\ldots,12$ and $n=1,2\ldots,16$. From the fitting results it is clear that the coefficients of the logarithmic correction term $d\ell=d$ with $\ell=1$ for all the cases, accurate up to at least the seventh decimal place. If we listed the values of ℓ from these fitting experiments in a table that is similar to Tables I or II, all the entries in the table would be 1. From these results we draw the conclusion that when the number of dimers d is fixed, there is a logarithmic correction term in the free energy per lattice site, and the coefficient of this term is exactly d, regardless of the parity of n, the width of the lattice strip.

The fitting results for ℓ along with other free parameters (c_1, c_2, c_3, and c_4) for $n=11$ and $n=12$ are shown in Tables V and VI, respectively.

IV. DISCUSSION

The close-packed dimer model, which is a special case of the general monomer-dimer model discussed here, already demonstrates intriguing connections with the theory of universality and finite scaling [14,15]. One of the peculiarities of the dimer model is the parity effect of the lattice strip width on the finite-size correction, which was already pointed out in the earlier studies (for example, Ref. [16]). This effect is attributed to the strong nonlocality of the dimer model. Recently, through mapping the close-packed dimer model to the

TABLE IV. Fitting $(mn)^{-1}\ln a_{(mn-v)/2}$ to Eq. (3) for $n=12$ with $b=1/2$ and c_0 fixed as $c_0^e(12)=0.279\,975\,752\,03$ given by Eq. (4). Only data with $m_0=100\leqslant m\leqslant 500$ are used in the fitting. Numbers in square brackets denote powers of 10.

v	c_1	c_2	c_3	c_4	ℓ
0	−0.154113	5.91813[−8]	−5.64565[−6]	0.000221538	1.13781[−9]
2	0.219858	−0.748376	−2.59592	−16.0843	1
4	0.535917	−2.62009	−9.74674	325.144	1.00012
6	0.818481	−5.66056	−12.3135	1291.39	1
8	1.07862	−10.0074	12.3026	2471.16	0.999318
10	1.32299	−15.8166	92.4989	2982.18	0.998001
12	1.55565	−23.1898	253.569	1852.54	0.996201
14	1.77871	−32.1427	512.746	-1715.1	0.994189
16	1.99286	−42.6063	878.094	-8235.31	0.99226
18	2.19782	−54.4469	1349.67	-17937.8	0.990678
20	2.39273	−67.4891	1921.53	-30816.5	0.989648
22	2.57645	−81.5376	2583.82	-46694.5	0.989309
24	2.74778	−96.3937	3324.52	-65284.5	0.989743

TABLE V. Fitting $(mn)^{-1} \ln a_d$ to Eq. (5) for $n=11$ with $c_0=0$. Only data with $m_0=100 \leqslant m \leqslant 500$ are used in the fitting.

d	c_1	c_2	c_3	c_4	ℓ
1	0.276775	−0.138528	0.0329753	−0.0340287	1
2	0.490536	−0.306122	0.0586802	−0.064429	1
3	0.667437	−0.502783	0.0713802	−0.0928228	1
4	0.818185	−0.728509	0.0653413	−0.122902	1
5	0.948647	−0.983302	0.0348304	−0.160464	1
6	1.06253	−1.26716	−0.0258842	−0.213442	1
7	1.16241	−1.58009	−0.122532	−0.291946	1
8	1.25014	−1.92208	−0.260842	−0.408293	1
9	1.32717	−2.29313	−0.446537	−0.577047	1
10	1.39462	−2.69326	−0.685341	−0.815052	1
11	1.4534	−3.12245	−0.98297	−1.14147	1
12	1.50428	−3.58071	−1.34514	−1.57782	1

spanning tree and the sandpile model, it is found that for a $m \times n$ lattice strip as m goes to infinity, changing the parity of n in essential changes the boundary condition of the model [15]. This change of boundary condition, however, is not obvious in the dimer model itself.

In this paper it is shown that for the more general monomer-dimer model with a fixed number of monomers v on a $m \times n$ lattice strip with free-boundaries, the finite-size correction of the free energy also depends on the parity of the width n. The coefficient of the logarithmic correction term of the free energy per lattice site is v when n is odd, and $v/2$ when n is even. As pointed out earlier, this result holds for any fixed number of monomers. As long as the number of monomers v is kept fixed as the length of the lattice m goes to infinity, the monomer density will approach zero. For different values of v, the free energy will approach the same limit as the close-packed dimer model with the same lattice width, as discussed previously for Eq. (4) and shown in Figures 1 and 2. On the other hand, for the low dimer density limit where the number of dimers d is fixed, the dimer density will approach zero as the length of the lattice strip m goes to infinity, and the free energy per lattice site will approach zero. In this limit, the coefficient of the logarithmic correction term equals to the number of dimers and does not depends on the parity of the lattice width n.

For the close-packed dimer model or when there is a single monomer on specific sites on the boundary of the lattice, there is a nice bijection between the dimer covering on the lattice and the spanning tree of a sublattice [7,14,15,21]. By using this one-to-one mapping, the hidden dependence of the boundary condition of the model on the parity of the lattice width can be revealed, and through the spanning tree problem the dimer model can be further linked to other well-studied models in statistical physics and graph theory, such as the Abelian sandpile model [14,15]. It would be nice to have such a bijection for the general monomer-dimer model discussed here. For the general monomer-dimer model with interior monomer(s), however, the bijection between the dimer covering and spanning tree does not hold. As pointed out earlier [7], there exist dimer configurations

TABLE VI. Fitting $(mn)^{-1} \ln a_d$ to Eq. (5) for $n=12$ with $c_0=0$. Only data with $m_0=100 \leqslant m \leqslant 500$ are used in the fitting.

d	c_1	c_2	c_3	c_4	ℓ
1	0.261291	−0.126812	0.0303175	−0.031145	1
2	0.46482	−0.27804	0.0548913	−0.0590459	1
3	0.63456	−0.453686	0.0692984	−0.0847911	1
4	0.780327	−0.653749	0.0691166	−0.11093	1
5	0.907498	−0.878229	0.0499239	−0.141493	1
6	1.01948	−1.12713	0.0072994	−0.182016	1
7	1.11861	−1.40044	−0.0631767	−0.239563	1
8	1.20661	−1.69817	−0.165923	−0.322746	1
9	1.2848	−2.02032	−0.305355	−0.441752	1
10	1.35421	−2.36689	−0.485889	−0.608364	1
11	1.41568	−2.73787	−0.711936	−0.835983	1
12	1.46989	−3.13327	−0.987907	−1.13966	1

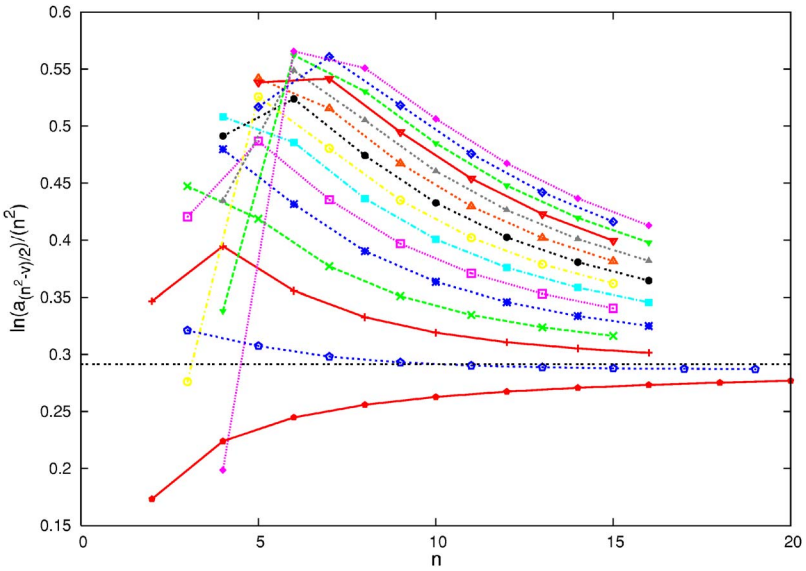

FIG. 3. (Color online) Free energy per lattice site $n^{-2}\ln(a_{(n^2-v)/2})$ on $n\times n$ square lattices as a function of n for different number of monomers $v=0,1,\ldots,14$. The curve for $v=0$ is on the bottom and that for $v=14$ is on the top. The dashed horizontal line is the value in the thermodynamic limit as $n\to\infty$: 0.291 560 904.

that cannot be mapped into spanning trees. New theoretical developments might be needed to interpret the findings reported here.

We mention briefly here that similar calculations have also been carried out for the monomer-dimer models on two-dimensional cylinder lattices. The results are similar to those reported here: In the low monomer density limit where the number of monomers is fixed, the coefficient of the logarithmic correction term of the free energy per lattice site depends on the number of monomers present (v) and the parity of the circumference n of the cylinder lattice: the coefficient equals to v when n is odd and $v/2$ when n is even. In the low dimer density limit where the number of dimers is fixed, the coefficient of the logarithmic correction term of the free energy per lattice site equals to the number of dimers, for both odd and even circumference n of the cylinder lattice.

APPENDIX: FREE ENERGY AND NUMBER-THEORETICAL PROPERTIES FOR FIXED NUMBER OF MONOMERS ON $n\times n$ SQUARE LATTICES

In this appendix, we investigate the free energy and number-theoretical properties of $a_{(n^2-v)/2}$, the number of ways to arrange $(n^2-v)/2$ dimers with fixed number v of monomers on $n\times n$ square lattices. The cases $v=0$ and $v=1$ are reported in Ref. [18].

In Figure 3 the free energy per lattice site $\ln(a_{(n^2-v)/2})/n^2$ on $n\times n$ square lattices as a function of n are shown for different number of monomers: $v=0,1,\ldots,14$. The curve for $v=0$ is on the bottom and that for $v=14$ is on the top. The dashed horizontal line is the value in the thermodynamic limit as $n\to\infty$: $G/\pi=0.291\ 560\ 904$. As the curves show, the close-packed case of $v=0$ stands out in the behavior of the free energy on $n\times n$ square lattices as compared to those cases where the number of monomers is not zero: when $v=0$, the free energy on the close-packed lattices approaches the thermodynamic limit monotonically from below. When $v\neq 0$, the free energy approaches the thermodynamic limit differently. The case of $v=1$ is already discussed in Ref. [18], where it is shown that as n increases, the free energy approaches and crosses the thermodynamic limit from above, reaches a minimum, and then approaches the thermodynamic limit from below. For $v>1$, Figure 3 shows that, after the initial irregularities for some small lattices (the size of which depends on v), the free energy decreases as n increases, as in the case of $v=1$. The current calculations cannot give a conclusive answer to the question whether those curves approach the thermodynamic limit monotonically from above, or they behave in a similar way as in the case of $v=1$ where they reach and pass the thermodynamic limit first, and then approach the thermodynamic limit from below.

One of the advantages of exact calculations is that the number-theoretical proprieties of the coefficients of the partition functions can be investigated. In Ref. [18], it is conjectured based on the exact calculations that when $v=1$ and $n=2k+1$, $a_{(n^2-1)/2}$ can be factored as $a_{(n^2-1)/2}=2^k c_k$ where c_k is an squarefree odd integer. Furthermore, $c_k \pmod{2^4}$ has a regular pattern. These properties are in parallel to those found for $v=0$ and $n=2k$, where exact solutions exist [20]. Here we investigate the number-theoretical proprieties of $a_{(n^2-1)/2}=2^k c_k$ for $v=2,3,\ldots,14$. For those values of v, unlike the situations when $v=0$ and $v=1$, there are some "irregularities" when the size of the lattices are small. These irregularities are evident in Figure 3. Due to these irregularities and the limited sizes of n in the calculations, there are not clear patterns of the factorizations of these numbers. However, except for a few initial irregularities when n is small, most of the time $a_{(n^2-v)/2}$ can be factored into 2^l (where l is a positive number), a few small prime numbers with low multiplicities (usually equal to 1), and a few big prime numbers. There is no clear pattern of $a_{(n^2-v)/2}/2^l \pmod{2^4}$ for $v>1$.

[1] P. W. Kasteleyn, Physica (Amsterdam) **27**, 1209 (1961); H. N. V. Temperley and M. E. Fisher, Philos. Mag. **6**, 1061 (1961); M. E. Fisher, Phys. Rev. **124**, 1664 (1961).
[2] J. F. Nagle, Phys. Rev. **152**, 190 (1966); D. Gaunt, *ibid.* **179**, 174 (1969).
[3] J. Bondy and D. Welsh, Proc. R. Soc., Math. Physic. Eng. Sci. **62**, 503 (1966); J. Hammersley, *ibid.* **64**, 455 (1968); J. Hammersley and V. Menon, J. Inst. Math. Appl. **6**, 341 (1970).
[4] M. E. Fisher and J. Stephenson, Phys. Rev. **132**, 1411 (1963); R. E. Hartwig, J. Math. Phys. **7**, 286 (1966).
[5] O. J. Heilmann and E. H. Lieb, Commun. Math. Phys. **25**, 190 (1972); C. Gruber and H. Kunz, *ibid.* **22**, 133 (1971).
[6] C. Kenyon, D. Randall, and A. Sinclair, J. Stat. Phys. **83**, 637 (1996).
[7] W.-J. Tzeng and F. Y. Wu, J. Stat. Phys. **110**, 671 (2003).
[8] P. W. Kasteleyn, J. Math. Phys. **4**, 287 (1963); M. E. Fisher, *ibid.* **7**, 1776 (1966).
[9] D. S. Rokhsar and S. A. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988); F. Alet, J. L. Jacobsen, G. Misguich, V. Pasquier, F. Mila, and M. Troyer, *ibid.* **94**, 235702 (2005).
[10] M. Jerrum, J. Stat. Phys. **48**, 121 (1987); **59**, 1087(E) (1990).
[11] M. R. Garey and D. S. Johnson, *Computers and Intractability, A Guide to the Theory of NP-Completeness* (W. H. Freeman and Company, New York, 1979). D. J. A. Welsh, *Complexity: Knots, Colourings, and Counting*, Vol. 186 of London Mathematical Society Lecture Note Series (Cambridge University Press, Cambridge, 1993).
[12] H. W. J. Blöte, J. L. Cardy, and M. P. Nightingale, Phys. Rev. Lett. **56**, 742 (1986); I. Affleck, *ibid.* **56**, 746 (1986).
[13] J. L. Cardy and I. Peschel, Nucl. Phys. **300[FS22]**, 377 (1988); V. Privman, Phys. Rev. B **38**, 9261 (1988).
[14] J. G. Brankov, J. Math. Phys. **36**, 5071 (1995).
[15] N. S. Izmailian, V. B. Priezzhev, P. Ruelle, and C.-K. Hu, Phys. Rev. Lett. **95**, 260602 (2005).
[16] A. E. Ferdinand, J. Math. Phys. **8**, 2332 (1967).
[17] C. Itzykson, H. Saleur, and J.-B. Zuber, Europhys. Lett. **2**, 91 (1986).
[18] Y. Kong, Phys. Rev. E **73**, 016106 (2006).
[19] Y. Kong, J. Chem. Phys. **111**, 4790 (1999).
[20] P. E. John and H. Sachs, Discrete Math. **216**, 211 (2000).
[21] H. N. V. Temperley, in *Combinatorics: Proceedings of the British Combinatorial Conference 1973* Vol. 13 of London Mathematical Society Lecture Note Series, (Cambridge University Press, Cambridge, England, 1974), pp. 202–204.